Maxwell's Equation

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Maxwell's Equation: Microscopic and Macroscopic

We know that, before Maxwell's their where following four equation/laws specifies divergence and curl of electric and magnetic field $E = B$ respectively.

1. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ Gauss law electrostatics (1)
2. $\nabla \cdot B = 0$ Gauss law in magnetostatics (2)
3. $\nabla \times E = -\frac{\partial B}{\partial t}$ Faraday's law (3)
4. $\nabla \times B = \mu_0 J$ Ampere's law (4)

These *equations* represent state of electromagnetic theory over a century ago,

Taking the divergence of equation (3) we have,

$$\nabla \cdot (\nabla \times E) = -\nabla \cdot \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot B) = 0$$

$$0 = 0$$

L.H.S. equals to zero, since, the divergence of curl is equal to zero and R.H.S. is equal to zero. According to Gauss law in magnetostatics.

Taking divergence of equation (4) we have,

$$\nabla \cdot (\nabla \times B) = \mu_0 J \qquad (5)$$

$$0 \neq 0$$

Here, L.H.S. equals to zero but R.H.S. is not equal to zero. The scientist Maxwell observed & used equation of continuity and Gauss law to correct Ampere's law,

The equation of continuity is given by,

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot J = -\frac{\partial}{\partial t}(\epsilon_0 \cdot \nabla \cdot E) \qquad \text{From } \nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot J = -\epsilon_0 \cdot \nabla \cdot \frac{\partial E}{\partial t}$$

$$\nabla \cdot J + \epsilon_0 \cdot \nabla \cdot \frac{\partial E}{\partial t} = 0$$

$$\nabla \cdot \left(J + \epsilon_0 \frac{\partial E}{\partial t}\right) = 0 \qquad (6)$$

$$\nabla \cdot (J + Jd) = 0$$

Where Jd displacement current density.

If we add $\epsilon_0 \frac{\partial E}{\partial t}$ with J in Ampere's law. We have,

$$\mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \qquad (7)$$

Such a modification changes as per magnetostatics is i.e. $E = constant$ we have equation (7) as curl $B = 0$

This equation is called Maxwell's correction to Ampere's law and it signifies "the change in electric field induces magnetic field."

Maxwell's called the extra term Jd as displacement current density given by,

$$Jd = \epsilon_0 \frac{\partial E}{\partial t} \qquad (8)$$

Thus, after correction by Maxwell we have set a fourth equation as,

- In electrostatics
  $\nabla \cdot E = \frac{\rho}{\epsilon_0}$
  The divergence of the electric field is equal to the charge density inside of a closed surface of interest, multiplied by a constant. The divergence is how much the electric field "spreads out" from a given point. If there is more charge inside, the divergence is greater. If it's zero, the divergence is zero.

- Gauss law in magnetostatics
  $\nabla \cdot B = 0$
  There are no magnetic monopoles. The divergence of B is always zero. As such, there is no "sink" or "source" for B - the field lines have no beginning and no end. There is no source for them like there is for an electric field (i.e. an electric monopole). All magnetic "charge" is found in a dipole, with a North and a South.

- Faraday's law
  $\nabla \times E = -\frac{\partial B}{\partial t}$
  The curl of the electric field is equal to the negative of the change in the magnetic field in time. In other words, if the magnetic field isn't changing, electric field lines are straight. If it is changing, the electric field "swirls" appropriately, depending on if the field is increasing or decreasing. A changing magnetic field can induce an electric field! (i.e. Faraday induction). The negative sign is called Lenz's law.

- Amperes law

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

The curl of B is equal to the Current Density (the amount of current per unit volume) plus any change in the electric field. This second part is often called the "displacement current, since it helps with dealing with capacitors.

These equations incorporate classical interaction between all electric charges and current in the system and are called Maxwell's microscopic equation.

Maxwell's macroscopic equation:

The macroscopic field equation provides correct classical picture for arbitrary field and towards distribution including both macroscopic and microscopic scales.

However for macroscopic substances it sometimes convenient to introduce new derived field which represent electric and magnetic field in which an average sense the material properties of substances are already included.

$$D = \epsilon_0 E \qquad (9)$$

$$\mu = \frac{B}{\mu_0} \qquad (10)$$

i.e. The derived fields are linearly proportional to primary fields and that electric displacement (magnetising field) is only dependent on the electric (magnetic) field.

The field equations expressed in terms of derived field quantity $D\ and\ H$ are

Microscopic		Macroscopic	
$\nabla \cdot E = \frac{\rho}{\epsilon_0}$	$\longrightarrow$	$\nabla \cdot E\epsilon_0 = \rho$	(1)
$\nabla \cdot B = 0$	$\longrightarrow$	$\nabla \cdot D = \rho$	(2)
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\longrightarrow$	$\nabla \times E = -\frac{\partial B}{\partial t}$	(3)
$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$	$\longrightarrow$	$\nabla \times B = \nabla \times \mu_0 H = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$	(4)

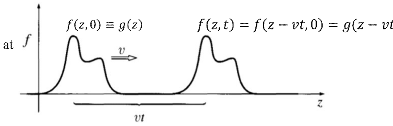
$$\nabla \times H = J + \frac{\partial D}{\partial t} = J + J_d$$

$$J_d = \frac{\partial D}{\partial t}$$

- Traveling wave

How to describe a wave that propagates with a fixed shape at constant speed mathematically?

Let $f(z,t)$ represent the displacement of the string at the point z, at time t.



Any function $f(z,t)$ that depends only on $z - vt$ represents a w[ave] of fixed shape traveling in the z direction at constant speed v.

$f_1(z,t) = Ae^{-b(z-vt)^2}$
$f_2(z,t) = A\sin[b(z-vt)]$
$f_3(z,t) = \dfrac{A}{b(z-vt)^2+1}$

represent travelling wave at constant speed v.

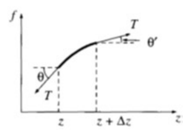
$f_4(z,t) = Ae^{-b(bz^2-vt)^2}$
$f_5(z,t) = A\sin bz\cos(bvt)^2$

do not represent travelling wave at constant speed v.

A and b are constants.

- The wave equation

Why does a stretched string support wave motion?



The net transverse force on the segment

$$\Delta F = T\sin\theta' - T\sin\theta$$

Assuming these angles are small, the sine can be replaced by the tangent:

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

Newton's second law says

$$\Delta F = ma = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

μ is the mass per unit length

$$T\frac{\partial^2 f}{\partial z^2}\Delta z = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} \qquad v = \sqrt{\frac{T}{\mu}}$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

The wave equation

Thank you